Comment on "Triggering Rogue Waves in Opposing Currents"

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The authors of a recent Letter [1] based their study of rogue waves in nonuniform currents on a modified nonlinear Schrödinger equation (NLSE; see Eq.(1) in [1]). However, I will show below that equation is not correct. It gives wrong solutions even in the first order on the supposedly small parameter $U/c_{\rm g}$, where $U(x)$ is a current, and $c_{\rm g} = g/(2\omega)$ [here ω is a mean frequency of a quasi-monochromatic wave train, and g is the gravity acceleration]. I also suggest an accurate variant of NLSE [see Eq.(6) below], valid in the presence of a large-scale nonuniform current under condition $(1 + 4\omega U/g) \gtrsim 0.2$.

The key point in the derivation is that with non-small $U/c_{\rm g}$ we may not assume a globally narrow wave-number spectrum, since a local wave-number k essentially [up to several times!] varies along x together with U [see Eq.(2) below]. First, we should find monochromatic solutions of the following linearized water-wave problem: $\eta_t = -(U\eta)_x + \hat{k}\psi$, and $-\psi_t = U\psi_x + g\eta$, where $\eta(x,t)$ is a vertical displacement of the free water surface from a steady-state profile, $\psi(x,t)$ is a surface value of the wave velocity potential, and a linear pseudo-differential operator $\hat{k}$ multiplies the Fourier image $\psi_k(t)$ by $|k|$. Assuming $\eta = \mathrm{Re}[Q(x,\omega)\exp(-i\omega t)]$ and $\psi = \mathrm{Re}[P(x,\omega)\exp(-i\omega t)]$, we have

$$\omega^2 P + i\omega(\partial_x U + U\partial_x)P - \partial_x U^2\partial_x P = g\hat{k}P. \qquad (1)$$

Since $U(x)$ changes over many typical wave-lengths, we can write an approximate solution of Eq.(1) in the form $P(x,\omega) \approx \Psi(x,\omega)\exp\left[i\int^x k(\omega,U)dx\right]$, where $\Psi(x,\omega)$ is a slowly varying function, and a positive function $k(\omega,U)$ satisfies the local dispersion relation for deep-water waves, $\omega = Uk + \sqrt{gk}$. In an explicit form,

$$k(\omega,U) = [g + 2\omega U - \sqrt{g^2 + 4g\omega U}]/(2U^2). \qquad (2)$$

It is important that with positive k we have $\hat{k}P \approx -i\partial_x P$. Substitution into Eq.(1) gives us the equation

$$\omega(U_x\Psi + 2U\Psi_x) - 2kU(U_x\Psi + U\Psi_x) - U^2 k_x\Psi + g\Psi_x \approx 0.$$

Multiplying it by Ψ and integrating on x, we obtain $[\omega U - kU^2 + g/2]\Psi^2 \approx \mathrm{const}$ (it is equivalent to the wave action conservation [2]), or $\Psi \approx -iC[1 + 4\omega U/g]^{-1/4}$, with a complex constant C. Since $gQ = i\omega P - UP_x$, we have $Q(x,\omega) = CM(x,\omega)\exp\left[i\int^x k(\omega,U)dx\right]$, where $M \approx (k/g)^{1/2}[1 + 4\omega U/g]^{-1/4}$. What is important, at small U the wave amplitude behaves as $M/M_0 \approx (1 - 2\omega U/g)$, while in [1] the corresponding factor is $\exp[-U/(2c_{\rm g})] \approx (1 - \omega U/g)$, which is not correct.

Let us now consider a linear superposition of the monochromatic solutions in a narrow frequency range near ω,

$$\begin{aligned}\eta &= \mathrm{Re}\int d\xi\tilde{C}(\xi)M(x,\omega+\xi)\, e^{\{-i(\omega+\xi)t + i\int^x k(\omega+\xi,U)dx\}} \\ &\approx \mathrm{Re}\big[\Theta(x,t)M(x,\omega)\, e^{\{-i\omega t + i\int^x k(\omega,U)dx\}}\big], \qquad (3)\end{aligned}$$

where $\Theta(x,t)$ is defined by the following integral,

$$\Theta(x,t) = \int d\xi\tilde{C}(\xi)\, e^{\{-i\xi t + i\int^x[k(\omega+\xi,U) - k(\omega,U)]dx\}}.$$

Since the frequency spectrum $\tilde{C}(\xi)$ is concentrated at small ξ in a range $\Delta\Omega \ll \omega$, we can expand $[k(\omega+\xi,U) - k(\omega,U)]$ in powers of ξ, and thus derive a partial-differential equation for $\Theta(x,t)$ in the linear regime, $-i\Theta_x = ik_\omega\Theta_t - (1/2)k_{\omega\omega}\Theta_{tt} + \cdots$. Here k_ω and $k_{\omega\omega}$ are partial derivatives of $k(\omega,U)$: $k_\omega = (4\omega/g)[1 + \upsilon + (1+\upsilon)^{1/2}]^{-1}$ and $k_{\omega\omega} = (2/g)(1+\upsilon)^{-3/2}$, with $\upsilon \equiv 4\omega U(x)/g$ [wave blocking takes place at $\upsilon_* = -1$].

Eq.(3) means that the wave envelope is $\tilde{A}(x,t) \approx \Theta(x,t)M(x,\omega)$ [$\tilde{A}$ here is not the same as A in [1], but $|\tilde{A}| = |A|$]. It results in the following estimate for the quantity $I = (k|A|)(k/\Delta K)$, where $\Delta K \approx k_\omega\Delta\Omega$ is a width of the local spectral k-distribution,

$$I_\upsilon/I_0 = 2^{\frac{3}{2}}[1 + \upsilon/2 + \sqrt{1+\upsilon}]^{-\frac{5}{2}}(\sqrt{1+\upsilon} + 1)(1+\upsilon)^{\frac{1}{4}}. \quad (4)$$

Thus, Eq.(8) in [1] should be corrected as written below,

$$A_{\max}(\upsilon)M_0/(M_\upsilon A_0) = 1 + 2\sqrt{1 - [\sqrt{2}\varepsilon N I_\upsilon/I_0]^{-2}}. \quad (5)$$

However, applicability of formula (4) implies $(\Delta\Omega)^2 \ll 6k_\omega/k_{\omega\omega\omega}$; otherwise $\Delta K \not\approx k_\omega\Delta\Omega$. Since practically important are values $\Delta\Omega \approx (0.1...0.2)\omega$, Eq.(4) can be good at $\upsilon \gtrsim -0.8$ only. The same condition arises when we require that the neglected linear higher-order dispersive terms are small comparatively to $k_{\omega\omega}\Theta_{tt}$.

To complete our derivation of NLSE with variable coefficients for a weakly nonlinear deep-water wave train in a large-scale nonuniform current, we have in a standard manner to take into account the nonlinear frequency shift, which for fixed k is well known to be $\delta\omega \approx \sqrt{gk}k^2|A|^2/2$. For fixed ω, that corresponds to the nonlinear wave-number shift $\delta k \approx -k_\omega\sqrt{gk}k^2|A|^2/2$. Using the relation $|A| \approx |\Theta|M$, we finally derive

$$i\Theta_x + ik_\omega\Theta_t - \frac{1}{2}k_{\omega\omega}\Theta_{tt} - \frac{k_\omega k^3\sqrt{k}}{2\sqrt{g + 4\omega U}}|\Theta|^2\Theta \approx 0. \quad (6)$$

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[1] M. Onorato, D. Proment, and A. Toffoli, Phys. Rev. Lett. **107**, 184502 (2011).

[2] F. P. Bretherton and C. J. R. Garrett, Proc. R. Soc. Lond. A **302**, 529 (1968).